Florida Department of Environmental Protection

Hydraulics 2: Problems on Conservation of Mass and Momentum

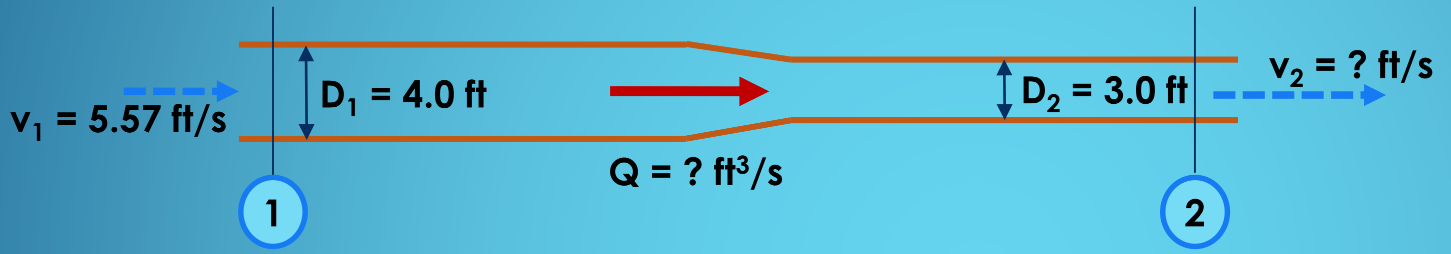
Mariano Guardo PhD, PE, D.WRE

April 29, 2015





- 1) A pipe 5000 ft long has a 48-in diameter an changes to 36-in at the middle of the reach. An average flow velocity of 5.57 ft/s has been measured in the 48-in section. Calculate the discharge and the average flow velocity through the 36-in diameter section



From Eq. 1.4:

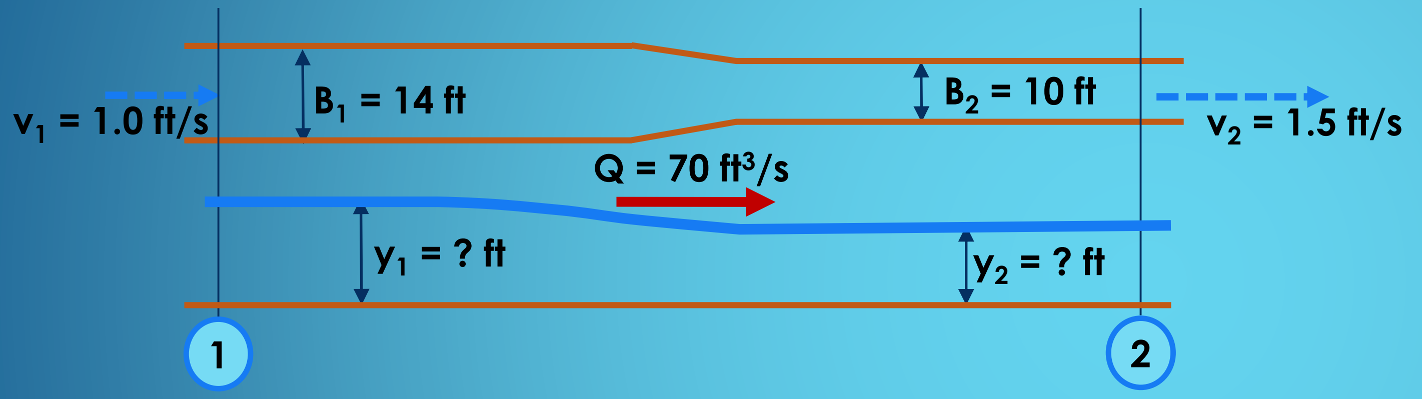
$$\boxed{Q = v_1 \times A_1 = v_2 \times A_2}$$

$A_1 = (\pi D_1^2/4) = \pi\,(4\text{ ft})^2/4 = 12.57\text{ ft}^2 \rightarrow$ **Q** $= 5.57\text{ ft/s} \times 12.57\text{ ft}^2 =$ **70 ft³/s**

$A_2 = (\pi D_2^2/4) = \pi\,(3\text{ ft})^2/4 = 7.07\text{ ft}^2 \rightarrow$ $\mathbf{v_2}$ $= (70\text{ ft}^3/\text{s}) / 7.07\text{ ft}^2 =$ **9.90 ft/s**

Conservation of Mass Problems

- 2) A canal 5000 ft long conveys 70 ft³/s with and average flow velocity of 1.0 ft/s. Its rectangular cross section is 14 ft wide. About half way in the reach, there is contraction and the cross section changes to 10 ft wide and the average flow velocity increases to 1.5 ft/s. Calculate the water depths upstream and downstream of the contraction



From Eq. 1.4:

$$\boxed{Q = v_1 \times A_1 = v_2 \times A_2}$$

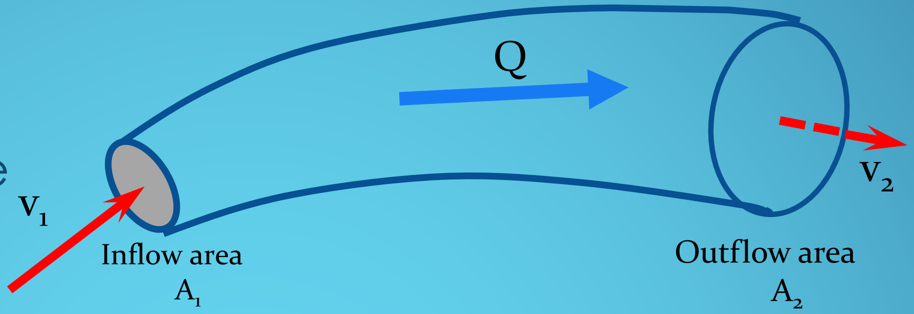
$A_1 = Q/v_1$ = 70 ft³/s / (1.0 ft/s) = 70 ft² → $A_1 = B_1 y_1$ → $\boldsymbol{y_1}$ = 70 ft²/14 ft = **5 ft**

$A_2 = Q/v_2$ = 70 ft³/s / (1.5 ft/s) = 46.67 ft² → $A_2 = B_2 y_2$ → $\boldsymbol{y_2}$ = 46.67 ft²/10 ft = **4.67 ft**

Conservation of Mass Problems

- The **Continuity Equation** as determined before

$$\boxed{Q = v_1 \times A_1 = v_2 \times A_2 \quad [1.4]}$$



represents steady flow for which the flow discharge remains constant between cross sections 1 (inflow) and 2 (outflow)

- When the flow discharges are different, the **Continuity Equation** becomes,

$$\boxed{Q_{in} - Q_{out} = \frac{\Delta S}{\Delta t} \quad [2.1]}$$

where $\boldsymbol{\Delta S/\Delta t}$ is the change in storage represented as $(S_{final} - S_{initial})/(t_f - t_i)$. The signs of the equation are perfectly established, for example if:
  1) $Q_{in} > Q_{out}$, an increase in storage will occur, so $S_{final} > S_{initial}$, and
  2) $Q_{in} < Q_{out}$, a decrease in storage will occur, so $S_{final} < S_{initial}$

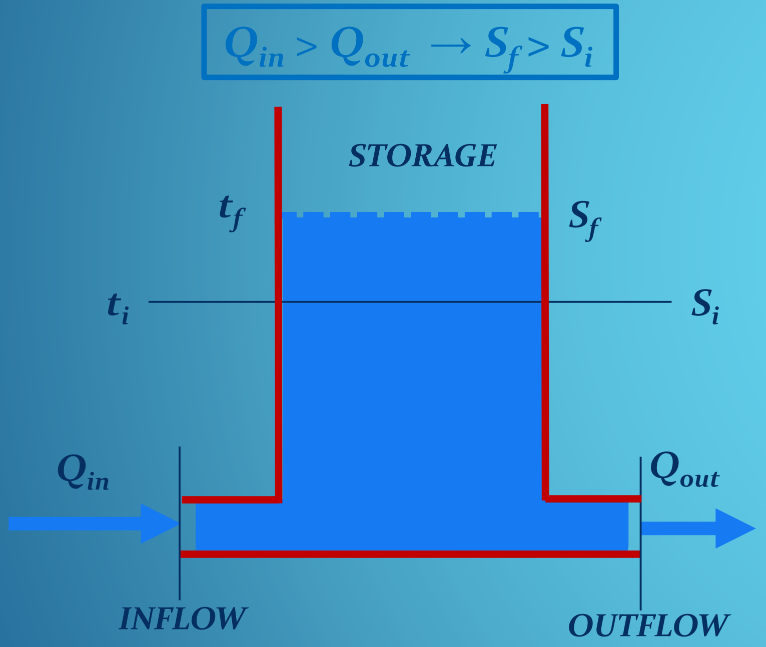
- Design of tanks is a typical application from this equation

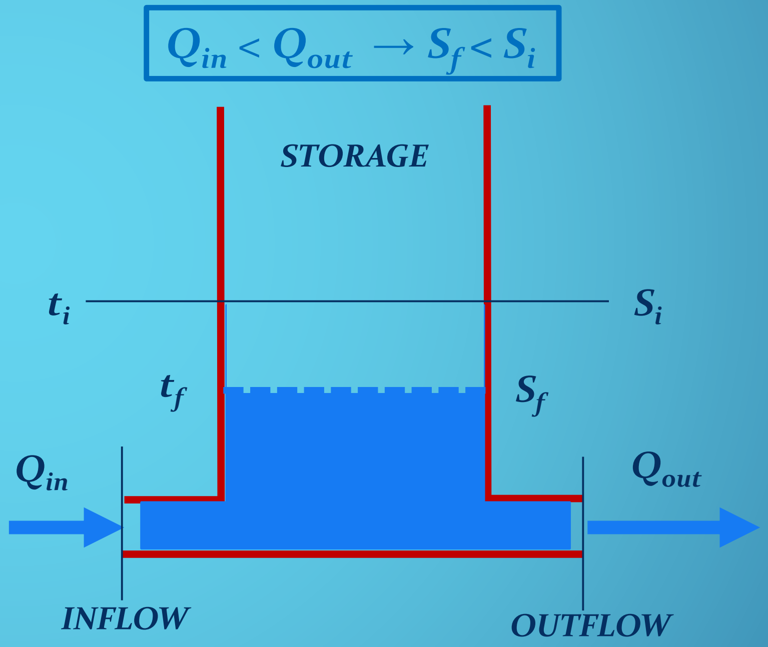
Conservation of Mass Problems

Conservation of Mass Problems

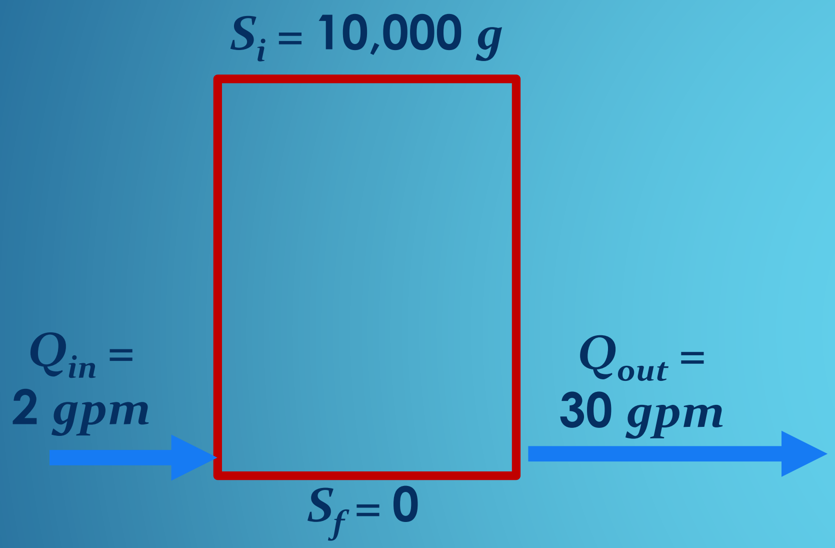
- The **Continuity Equation** when the flow discharges are different is represented by a control volume for a time interval t, from t_i to t_f, which accounts for the change in storage

$$\boxed{Q_{in} - Q_{out} = \frac{S_f - S_i}{t_f - t_i} \quad [2.2]}$$





- 3) A water supply system for a farm has a tank of 10,000 gallons. The inflow to the tank is provided by a well with pumping capacity of 2 gpm. When the tank is full, how long would it take to empty it out if its supply discharge is 30 gpm?



From Eq. 2.2:

$$\boxed{Q_{in} - Q_{out} = \frac{S_f - S_i}{t}}$$

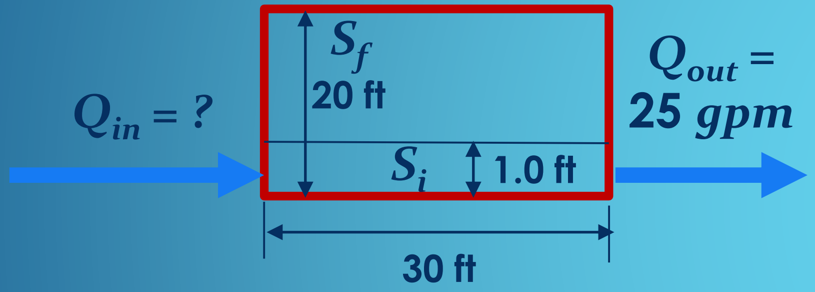
2 gpm – 30 gpm = (0 – 10,000 g)/t $\rightarrow$ t = – 10,000 g/ –28 gpm = 357.14 min

t = 5.95 hr = **5 hr 57 min**

Conservation of Mass Problems

Conservation of Mass Problems

- 4) A small community has a tank 30 by 30 ft (square base) and 20 ft high to full capacity. When the water depth in the tank is one foot, the inflow pump has to start pumping to fill it up. The continuous supply from the tank is 25 gpm. It has been estimated that the tank has to be refilled to maximum capacity in 2 days. Compute the discharge of the inflow pump to the tank, and the supply emergency time of the tank (i.e., w/o receiving inflow)?



From Eq. 2.2:

$$\boxed{Q_{in} - Q_{out} = \frac{S_f - S_i}{t}}$$

7.48052 g = 1.0 ft^3

S_i = (30 ftx30 ft)x1.0 ft=900 ft^3

S_f = (30 ftx30 ft)x20 ft=18,000 ft^3

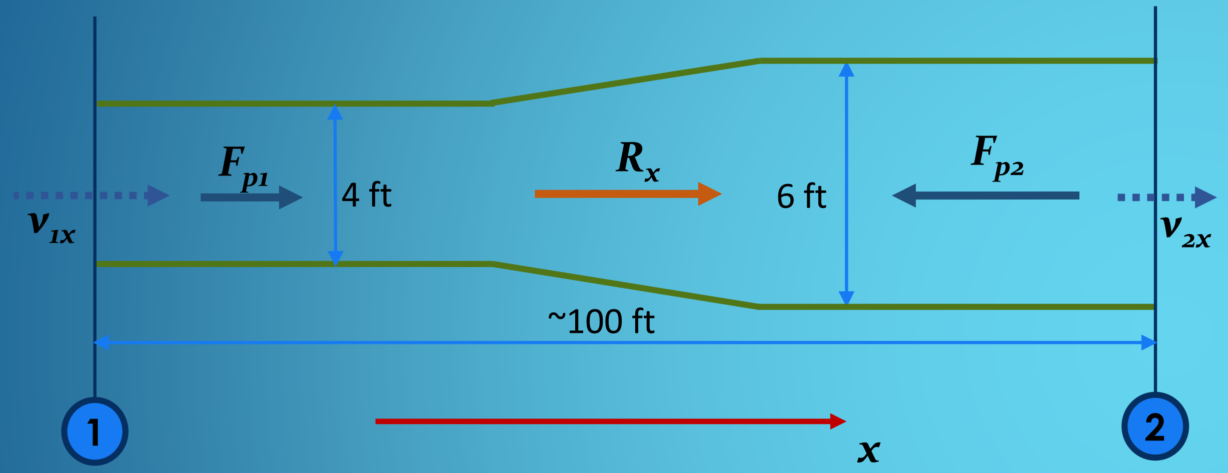
Q_{in} – 25 gpm = (134,649.36 – 6732.47)g / (2 d x 1440 min/d)

Q_{in} = 25 gpm + 44.42 gpm = **69.42 gpm**

0 – 25 gpm = (6732.47 – 134,649.36)g / t → t = –127,916.89 g / –25 gpm =

t = 5116.68 min / 1440 min/d = 3.55 d = **3 d 13.3 h**

- 5) Compute the hydrodynamic force produced by a flow discharge of 1100 ft³/s through an expansion from 4-foot to 6-foot diameter. Pressures upstream and downstream are 100 and 124.85 psi, respectively. The pipes do not have any slope (z = 0).



From Eq. 1.8:

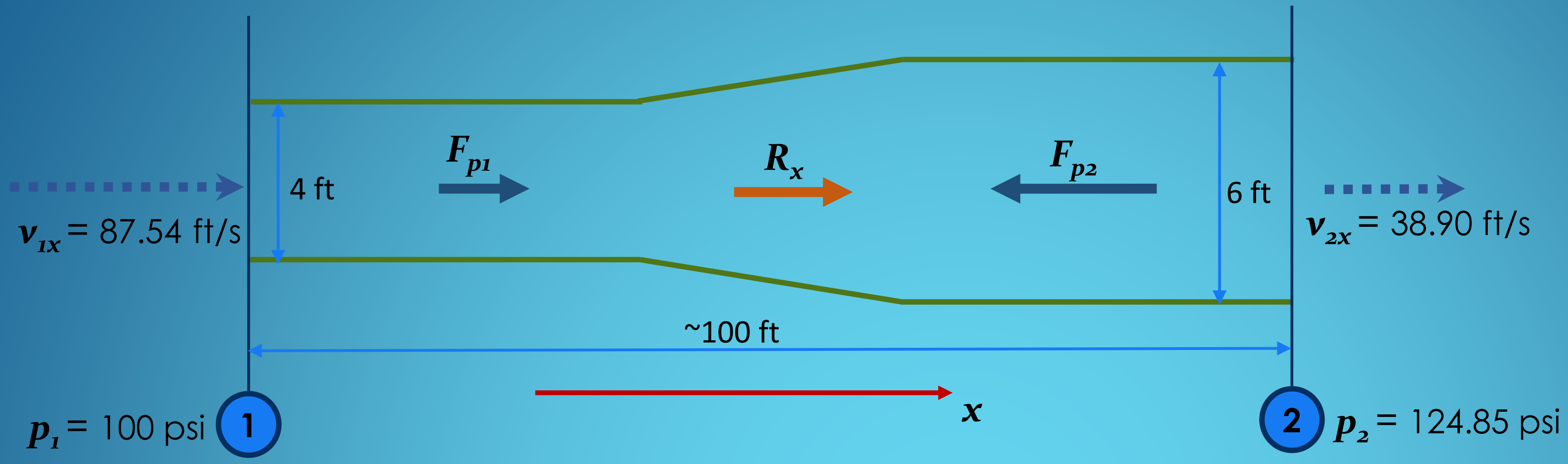
$$\boxed{\Sigma F_x = \rho Q\left(v_{2x} - v_{1x}\right)}$$

$$F_{p1} + R_x - F_{p2} = 1.94 \text{ lb-s}^2/\text{ft}^4 (1100 \text{ ft}^3/\text{s})(v_{2x} - v_{1x})$$

$A_1 = \pi\ (4 \text{ ft})^2/4 = 12.57 \text{ ft}^2$ $\quad v_{1x} = (1100 \text{ ft}^3/\text{s})\ /12.57 \text{ ft}^2 = 87.54 \text{ ft/s}$

$A_2 = \pi\ (6 \text{ ft})^2/4 = 28.27 \text{ ft}^2$ $\quad v_{2x} = (1100 \text{ ft}^3/\text{s})\ /28.27 \text{ ft}^2 = 38.90 \text{ ft/s}$

Conservation of Momentum Problems



$F_{p1} + R_x - F_{p2} = 1.94 \text{ lb-s}^2/\text{ft}^4 (1100 \text{ ft}^3/\text{s})(38.90 - 87.54) \text{ ft/s}$

$F_{p1} + R_x - F_{p2} = -103{,}510 \text{ lb}$

From Eq. 1.10: $F_{p1} = 100 \text{ lb/in}^2 (144 \text{ in}^2/\text{ft}^2)(12.57 \text{ ft}^2) = 180{,}956 \text{ lb}$

$F_{p2} = 124.85 \text{ lb/in}^2 (144 \text{ in}^2/\text{ft}^2)(28.27 \text{ ft}^2) = 508{,}327 \text{ lb}$

$180{,}956 \text{ lb} + R_x - 508{,}327 \text{ lb} = -103{,}510 \text{ lb}$

$R_x = 508{,}327 \text{ lb} - 180{,}956 \text{ lb} - 103{,}510 \text{ lb} = 223{,}861 \text{ lb}$

$HF_x = \mathbf{-223{,}861 \text{ lb}}$

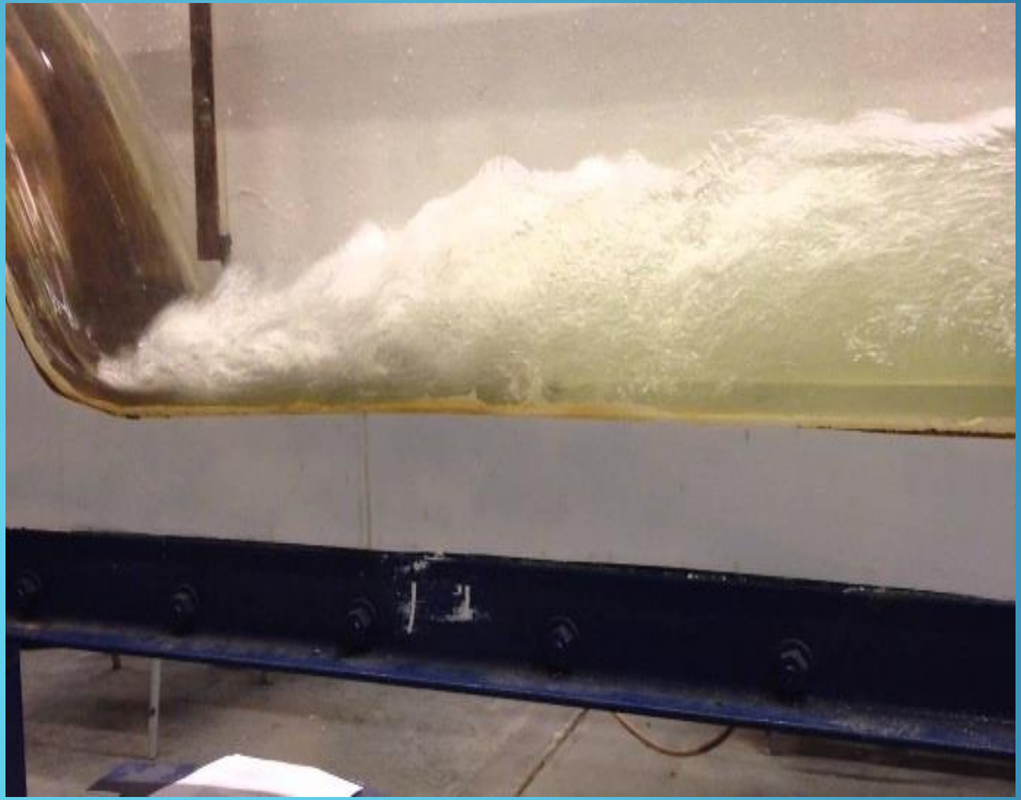
Conservation of Momentum Problems

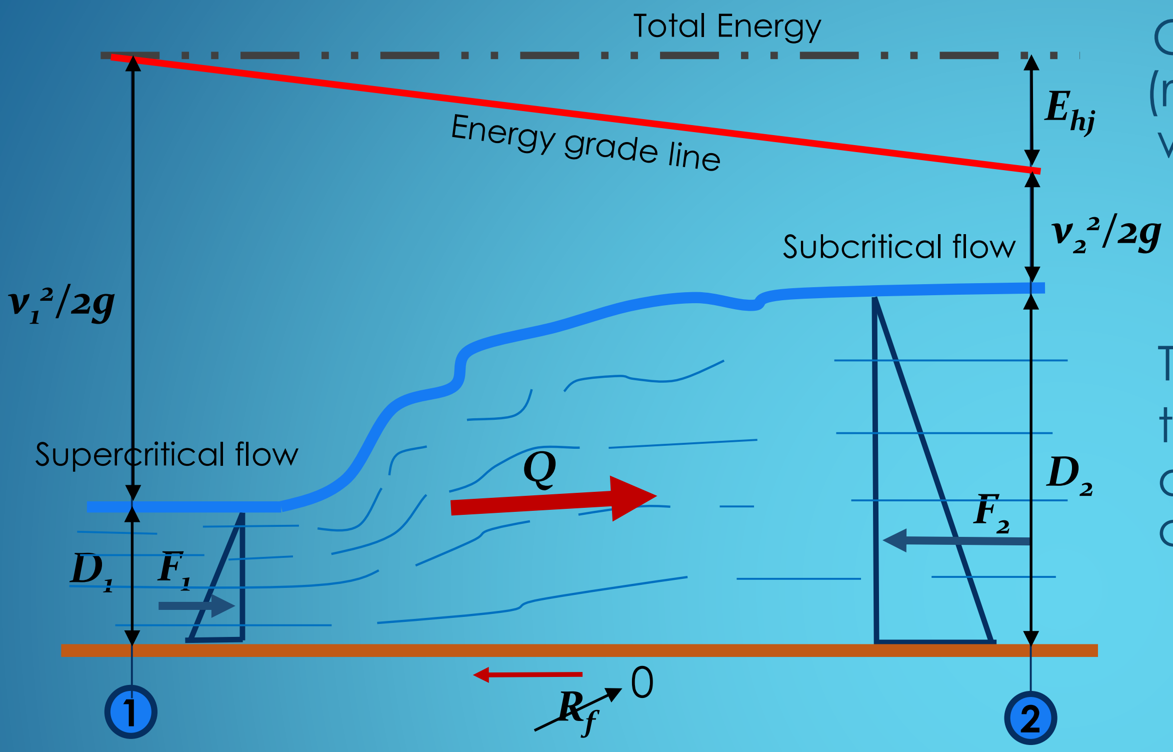
- An important application from the Conservation of Momentum in Hydraulics is the ***Hydraulic Jump***. This is caused by an instantaneous change in flow regime from supercritical to subcritical flow. It typically occurs in spillways of large dams when large flow velocities are generated through the chute (large kinetic energy) and an energy dissipator may be required in addition to the energy dissipated by the hydraulic jump

The Hydraulic Jump

The Hydraulic Jump

- Transition from supercritical to subcritical flow regime is not possible by gradually retarding the flow conditions. The physical contradictions can be explained by the concept of specific energy and critical depth (at minimum energy). This bears the physical properties of the ***Hydraulic Jump*** and the fact that it cannot be solved by the energy (Bernoulli) equation





Conservation of Momentum (rectangular channel with width = B)

$$\Sigma F_x = \rho Q\left(v_{2x} - v_{1x}\right) \quad [1.8]$$

The hydrostatic forces are of the form γdA, where d is the depth to the centroid of the cross section and $\rho = \gamma/g$

$$F_1 - F_2 = \rho Q\left(v_2 - v_1\right)$$

$\gamma\,(D_1/2)\,(D_1B) - \gamma\,(D_2/2)\,(D_2B) = \gamma/g\;Q\,[Q/(D_2B) - Q/(D_1B)]$ dividing by B

$D_1^2/2 - D_2^2/2 = (Q/B)^2\,1/(gD_2) - (Q/B)^2\,1/(gD_1)$ unit flow $q = Q/B$

$D_1^2/2 + q^2/(gD_1) = D_2^2/2 + q^2/(gD_2)$

By arithmetic manipulation and rearranging terms, this equation becomes

The Hydraulic Jump

$$D_1^{\ 2} + D_2 D_1 - \frac{2q^2}{gD_2} = 0$$

- D_1 and D_2 are known as the conjugate depths of the ***Hydraulic Jump***. Solving the quadratic yields

$$\boxed{D_1 = \frac{D_2}{2}\left[-1+\sqrt{1+8F_2^{\ 2}}\right] \quad [2.3]}$$ or $$\boxed{D_2 = \frac{D_1}{2}\left[-1+\sqrt{1+8F_1^{\ 2}}\right] \quad [2.4]}$$

where $F^2 = v^2/(gD) = q^2/(gD^3)$

- The head (Energy) loss in the ***Hydraulic Jump*** is,

$$\boxed{E_{hj} = \frac{(D_2 - D_1)^3}{4D_1 D_2} \quad [2.5]}$$

- There is a unique relation between the conjugate depths and the flow

$$\boxed{q = \sqrt{(gD_1 D_2)\frac{D_1 + D_2}{2}} \quad [2.6]}$$

The Hydraulic Jump

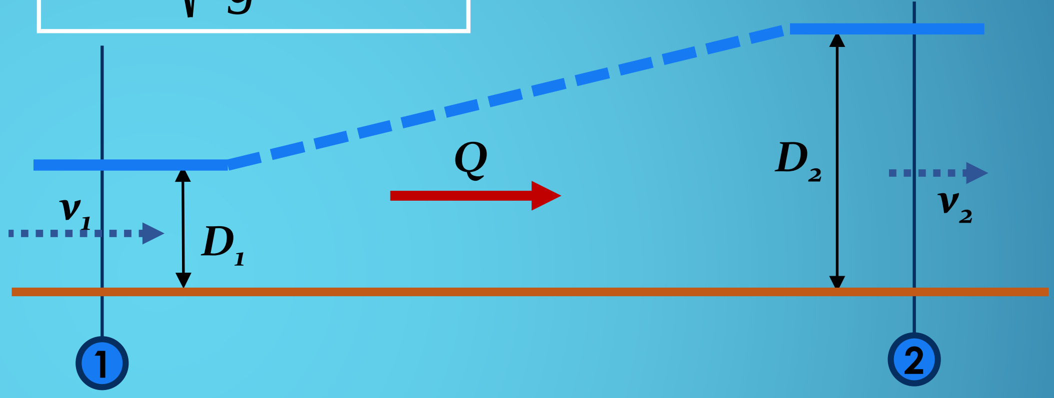
- 6) A rectangular channel 20 ft wide carries a flow discharge of 900 ft³/s. At a certain cross section its average flow velocity is 6 ft/s

  6.1) Is the flow at that cross section subcritical or supercritical?

  6.2) If a ***HJ*** occurred, what would be the conjugate depth?

  6.3) Calculate the head (Energy) loss in the ***HJ***

$$D_c = \sqrt[3]{\frac{q^2}{g}} \quad [2.7]$$



$q = Q/B$ = (900 ft³/s) / 20 ft = 45 ft³/s/ft

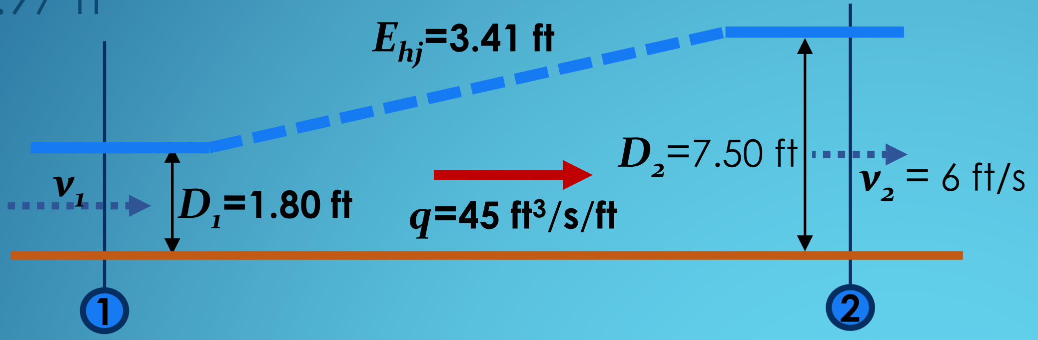
From Eq. 2.7,

D_c = [(45 ft³/s/ft)² / 32.17 ft/s²]$^{1/3}$ = **3.97 ft**

$A = Q/v$ = (900 ft³/s)/6 ft/s = 150 ft² → $D = A/B$ = (150 ft²) / 20 ft = **7.5 ft**, so there is ***subcritical*** flow because D_2 = 7.5 ft > D_c

Conservation of Momentum Problems

D_c = 3.97 ft



$$\boxed{D_1 = \frac{D_2}{2}\left[-1+\sqrt{1+8F_2^{\,2}}\right] \quad [2.3]}$$

From Eq. 2.3, the conjugate D_1 can be calculated. But F_2 needs to be calculated: F_2 = (6 ft/s)/[32.17 ft/s² x 7.5 ft]$^{1/2}$ = **0.39** < 1 ***subcritical*** √

$$D_1 = \frac{7.5\ ft}{2}\left[-1+\sqrt{1+8(0.39)^2}\right] = 3.75\times 0.48\ ft = 1.80\ ft$$

From Eq. 2.5,

$$E_{hj} = \frac{(D_2-D_1)^3}{4D_1D_2} = \frac{(7.5-1.8)^3\ ft^3}{4\times 7.5\ ft\times 1.8\ ft} = 3.41\,ft$$

From Eq. 2.6 (check),

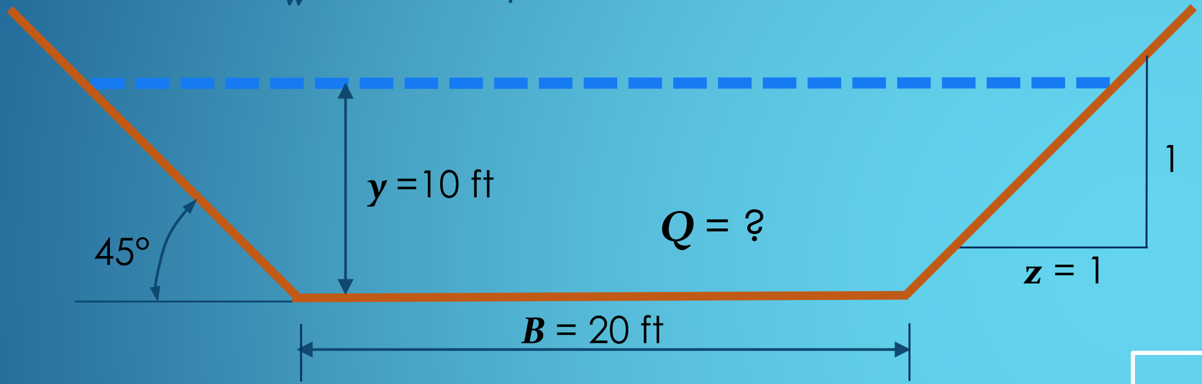
$$q = \sqrt{(gD_1D_2)\frac{D_1+D_2}{2}} = \sqrt{32.17\ ft/s^2\times 1.8\ ft\times 7.5\ ft\,\frac{1.8\ ft+7.5\ ft}{2}} = 45\ ft^2/s$$

Conservation of Momentum Problems

- 7) Compute the uniform flow discharge for a trapezoidal canal with water depth, y = 10 ft; side slope, z = 1; bottom width, B = 20 ft and invert slope, S = 1 ft/mile. The earth canal excavated in alluvial soil and has growth of grass

$\boldsymbol{n}$ = 0.028 $\boldsymbol{S}$ = 1 ft/mi = 1 ft/5280 ft = 0.0001894

$\boldsymbol{A}$ and $\boldsymbol{P_w}$ for a trapezoidal cross section can be calculated as follows,



$$A = (B + zy) \times y \quad [2.8]$$

$$P_w = B + 2y\sqrt{1+z^2} \quad [2.9]$$

$$Q = \frac{1.49}{n} A R_h^{2/3} \sqrt{S} \quad [1.16]$$

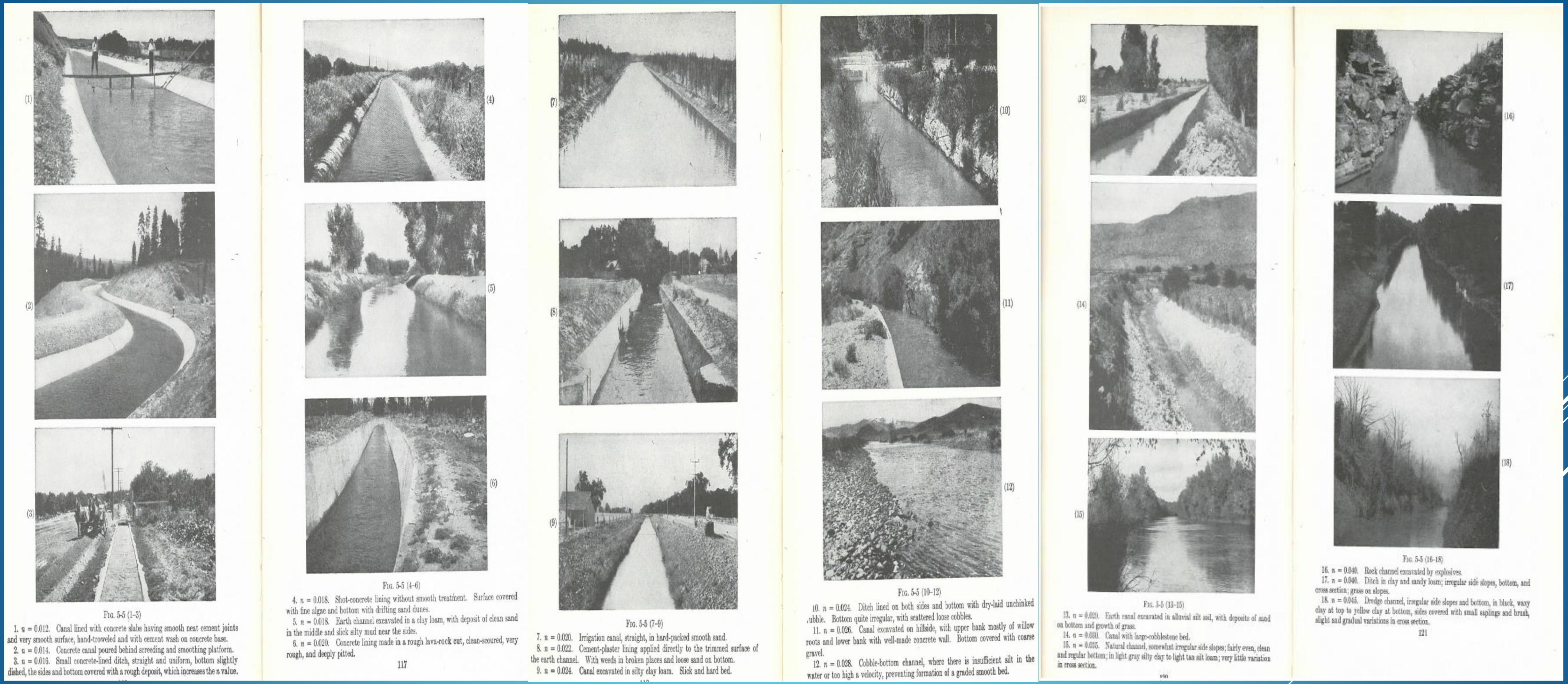
From ***Manning Equation*** (Eq. 1.16),

$\boldsymbol{A}$ = (20 ft + 1x10 ft) 10ft = 300 ft^2

$\boldsymbol{P_w}$ = 20 ft + 2x10 ft $(1+1^2)^{1/2}$ = 48.28 ft → $\boldsymbol{R_h}$ = 300 ft^2/48.28 ft = 6.21 ft

$$Q = \frac{1.49}{0.028}(300\, ft^2)(6.21\, ft)^{2/3}\sqrt{0.0001894} = 742.5\, ft^3/s$$

Uniform Flow Problem (Manning)



(1) (2) (3)

Fig. 5-5 (1-3)

1. $n = 0.012$. Canal lined with concrete slabs having smooth neat cement joints and very smooth surface, hand-troweled and with cement wash on concrete base.
2. $n = 0.014$. Concrete canal poured behind screeding and smoothing platform.
3. $n = 0.016$. Small concrete-lined ditch, straight and uniform, bottom slightly dished, the sides and bottom covered with a rough deposit, which increases the n value.

(4) (5) (6)

Fig. 5-5 (4-6)

4. $n = 0.018$. Shot-concrete lining without smooth treatment. Surface covered with fine algae and bottom with drifting sand dunes.
5. $n = 0.018$. Earth channel excavated in a clay loam, with deposit of clean sand in the middle and slick silty mud near the sides.
6. $n = 0.020$. Concrete lining made in a rough lava-rock cut, clean-scoured, very rough, and deeply pitted.

117

(7) (8) (9)

Fig. 5-5 (7-9)

7. $n = 0.020$. Irrigation canal, straight, in hard-packed smooth sand.
8. $n = 0.022$. Cement-plaster lining applied directly to the trimmed surface of the earth channel. With weeds in broken places and loose sand on bottom.
9. $n = 0.024$. Canal excavated in silty clay loam. Slick and hard bed.

(10) (11) (12)

Fig. 5-5 (10-12)

10. $n = 0.024$. Ditch lined on both sides and bottom with dry-laid unchinked rubble. Bottom quite irregular, with scattered loose cobbles.
11. $n = 0.026$. Canal excavated on hillside, with upper bank mostly of willow roots and lower bank with well-made concrete wall. Bottom covered with coarse gravel.
12. $n = 0.028$. Cobble-bottom channel, where there is insufficient silt in the water or too high a velocity, preventing formation of a graded smooth bed.

(13) (14) (15)

Fig. 5-5 (13-15)

13. $n = 0.029$. Earth canal excavated in alluvial silt soil, with deposits of sand on bottom and growth of grass.
14. $n = 0.030$. Canal with large-cobblestone bed.
15. $n = 0.035$. Natural channel, somewhat irregular side slopes; fairly even, clean and regular bottom; in light gray silty clay to light tan silt loam; very little variation in cross section.

(16) (17) (18)

Fig. 5-5 (16-18)

16. $n = 0.040$. Rock channel excavated by explosives.
17. $n = 0.040$. Ditch in clay and sandy loam; irregular side slopes, bottom, and cross section; grass on slopes.
18. $n = 0.045$. Dredge channel, irregular side slopes and bottom, in black, waxy clay at top to yellow clay at bottom, sides covered with small saplings and brush, slight and gradual variations in cross section.

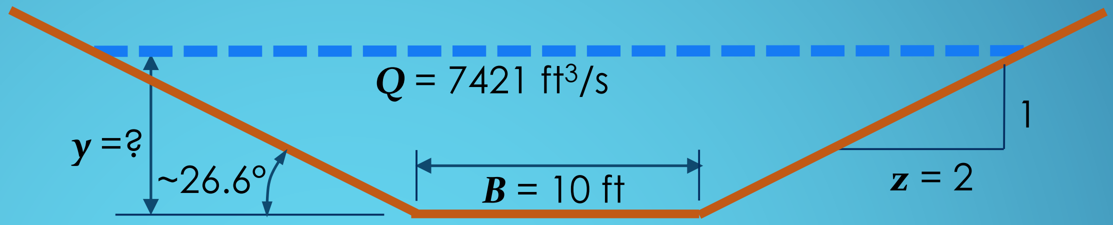
121

Typical n Values for Different Channels

- 8) A trapezoidal canal has the following design characteristics: Q = 7421 ft^3/s, z = 2 (side slope 2:1), B = 10 ft and n = 0.016 (unfinished concrete). Assuming uniform flow, compute the water depth for

  8.1) S = 0.00730

  8.2) S = 0.00055



Manning Equation (Eq. 1.16) can be rewritten as,

$$\boxed{\frac{Q \times n}{1.49 \times \sqrt{S}} = AR_h^{2/3}}$$

Terms on the left side of the equation are known, and the terms on the right side of the equation contain the unknown, which is y (water depth)

$$\frac{7421\,ft^3/s \times 0.016}{1.49 \times \sqrt{0.0073}} = 932.6844 = [(B+zy)y]\frac{[(B+zy)y]^{2/3}}{[B+2y\sqrt{1+z^2}]^{2/3}}$$

Uniform Flow Problem (Manning)

The right side of the previous equation can be developed based on Eqs. 2.8 and 2.9,

$$932.6844 = [(10+2y)y]\frac{[(10+2y)y]^{2/3}}{[10+2y\sqrt{1+2^2}]^{2/3}} = \frac{[(10+2y)y]^{5/3}}{[10+4.4721\times y]^{2/3}}$$

The above equation does not have a direct solution so it can be solved by trial and error,

For S = 0.00730 → Left side of Equation = 932.6844 → y = **10 ft**

For S = 0.00055 → Left side of Equation = 3397.9329 → y = **17.52 ft**

	Known Variables				
	Q (ft^3/s) =	**7421**	**S =**	**0.0073**	
	n =	**0.016**			
	B (ft) =	**10**			
	z =	**2**			
		932.6844	=[(10+2y)y]$^{5/3}$/ [10+4.4721*y]$^{2/3}$		
		Convergence		**Solution**	
y =	9.99969	932.6831		**10.00**	

	Known Variables				
	Q (ft^3/s) =	**7421**	**S =**	**0.00055**	
	n =	**0.016**			
	B (ft) =	**10**			
	z =	**2**			
		3397.9329	=[(10+2y)y]$^{5/3}$/ [10+4.4721*y]$^{2/3}$		
		Convergence		**Solution**	
y =	17.52233	3397.9281		**17.52**	

Uniform Flow Problem (Manning)

Uniform Flow Problem (R_h)

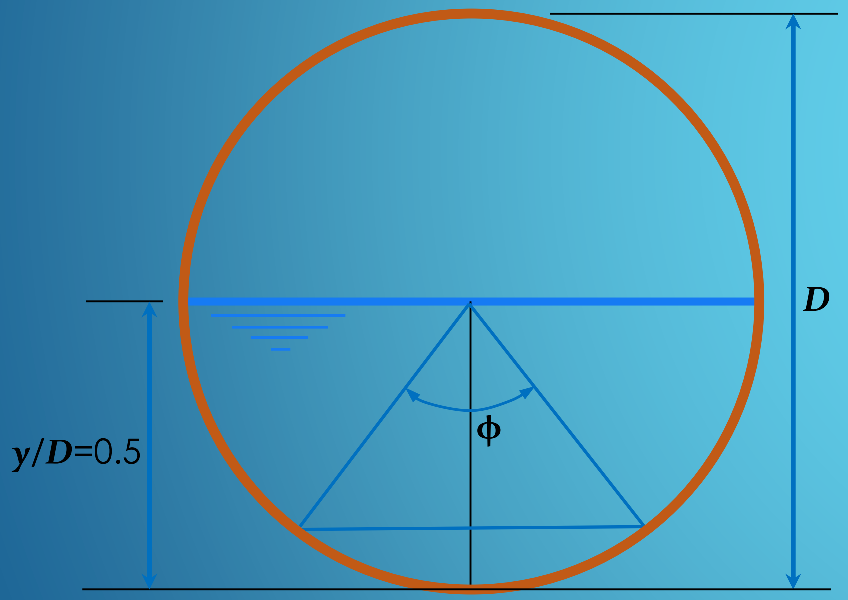
- 9) Develop a generic (and dimensionless) function for the R_h of a pipe, and plot the function as follows,

    9.1) Express y (water depth) on y-axis as y/D every 0.10

    9.2) $A = 1/8\ (\phi - \sin\phi)\ D^2$, where ϕ = angle (radians) and D = diameter

    9.3) $P_w = 1/2\ \phi\ D$

    9.4) $R_h = A/P_w = 1/4\ [(\phi - \sin\phi)\ /\ \phi]\ D$. Express R_h/D (dimensionless) on x-axis as $1/4\ [(\phi - \sin\phi)\ /\ \phi]$



Use the plot to determine the R_h of a 48-in pipe when the flow is both full and half-full. Determine at which water depth would occur the maximum R_h

Hint: $\phi = 2\ \arccos\ [(0.5 - (y/D))\ /\ 0.5]$
from 0° ($y/D = 0$) to 360° ($y/D = 1$)
for $y/D = 0 \rightarrow R_h/D = 0$

$2 \arccos[(0.5 - (y/D)) / 0.5]$

$1/8\,(\phi - \sin\phi)$

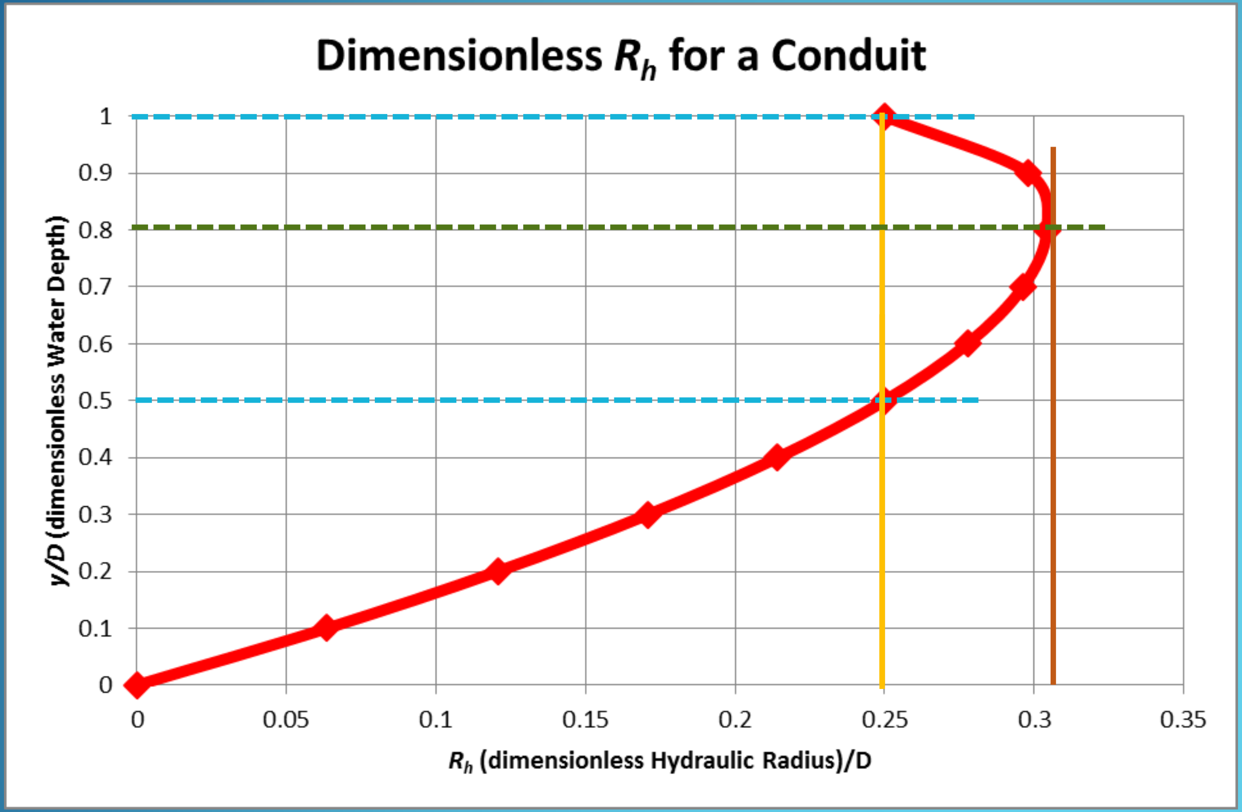
$1/2\,\phi$

$1/4\,[(\phi - \sin\phi)/\phi]$

Dimensionless Hydraulic/Geometric Functions of a Conduit

y/D	ϕ (°)	A/D^2	P_w/D	R_h/D
0.0	0.00	0.0000	0.0000	0.0000
0.1	73.74	0.0409	0.6435	0.0635
0.2	106.26	0.1118	0.9273	0.1206
0.3	132.84	0.1982	1.1593	0.1709
0.4	156.93	0.2934	1.3694	0.2142
0.5	180.00	0.3927	1.5708	0.2500
0.6	203.07	0.4920	1.7722	0.2776
0.7	227.16	0.5872	1.9823	0.2962
0.8	253.74	0.6736	2.2143	0.3042
0.9	286.26	0.7445	2.4981	0.2980
1.0	360.00	0.7854	3.1416	0.2500

Uniform Flow Problem (R_h)



For D = 4-foot (48-in) → half-flow y/D = 0.5 → y = **2.0 ft**; R_h/D = 0.25 → R_h = **1.0 ft**

full-flow y/D = 1.0 → y = **4.0 ft**; R_h/D = 0.25 → R_h = **1.0 ft**

Maximum R_h/D = 0.3043 → y/D = 0.81, so Max R_h = **1.22 ft** for y = **3.24 ft**

Uniform Flow Problem (R_h)

Florida Department of Environmental Protection

Questions?

Thank you!